TORSION IN AN INCOMPLETE TORE

W. F. CALLAWAY

TORSION IN AN INCOMPLETE TORE

An approximate solution for the
stress distribution in a circu-
lar ring sector under uniform
torsion using energy methods

by

William Franklin Callaway
Lieutenant Commander, United States Navy

Submitted in partial fulfillment
of the requirements
for the degree of
MASTER OF SCIENCE
IN MECHANICAL ENGINEERING

United States Naval Postgraduate School
Monterey, California
1952

This work is accepted as fulfilling
the thesis requirements for the degree of
MASTER OF SCIENCE
in
MECHANICAL ENGINEERING

from the
United States Naval Postgraduate School

Chairman
Department of Mechanical Engineering

Approved:

Academic Dean

ACKNOWLEDGMENT

The author desires to express his grateful appreciation for the guidance and encouragement given by Professor Robert Newton, U. S. Naval Postgraduate School, during the preparation of this work.

Monterey, California

June 1952

TABLE OF CONTENTS

INTRODUCTION

The stress distribution in an incomplete tore loaded as shown in Fig. 1 is of particular interest since it very closely approximates that in heavy close-coiled helical springs under axial tension or compression. Necessarily the spring helix angle must be small, which is the case in a close-coiled spring. By a heavy spring is meant one whose ratio of mean diameter to cross-sectional diameter is such a value that the curvature of the section must be considered.

It should be noted that the stress distribution arising from the loading in Fig 1. is not pure torsion in the usual sense, but is a combination of torsion and direct shear. The problem therefore resolves itself into one of finding a single stress function which defines the true stress distribution in a cross-section of the circular ring sector.

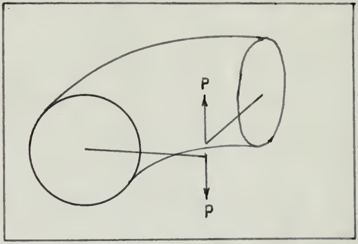

Fig. 1.

Several solutions to the problem are in the literature, all of which by various means solve the differential equation arising from the conditions of compatibility. The first, by Michell (1) in 1899, used polynomial stress functions and obtained solutions for approximately circular cross-sections. Göhner (2) used successive approximations to approach an exact solution. Shepherd (3) used a method similar to both Göhner and Michell by finding a sequence of functions for approximately circular cross-sections and combining them linearly in such a manner that the sum was a solution.

INTRODUCTION

The stress distribution in an incomplete torus loaded as shown in Fig. 1 is of particular interest since it very closely approximates that in heavy close-coiled helical springs under axial tension or compression. Essentially the spring helix angle must be small, which is the case in a close-coiled spring. In a heavy spring in which the index ratio of mean diameter to cross-sectional diameter is such a value that the curvature of the coil must be considered.

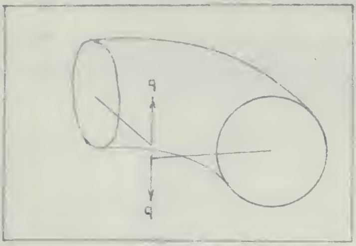

Fig. 1.

It should be noted that the stress distribution arising from the loading in Fig 1, is not pure torsion in the usual sense, but is a combination of torsion and direct shear. The problem therefore resolves itself into one of finding a single stress function which defines the true stress distribution in a cross-section of the circular ring sector.

Several solutions to the problem are in the literature, all of which by various means solve the differential equation arising from the conditions of compatibility. The first, by Michell (1) in 1899, used polynomial stress functions and obtained solutions for approximately circular cross-sections. Gohner (2) used successive approximations to approach an exact solution. Shepherd (3) used a method similar to both Gohner and Michell by finding a sequence of functions for approximately circular cross-sections and combining them linearly in such a manner that the sum was a solution.

Wahl (4) obtained a solution using curved bar theory and assuming a displacement of the center of rotation. Southwell (5) presented a formal solution for an arbitrary cross-section with a view towards a "relaxation" approach. Frieberger (6) has presented an exact solution for a circular cross-section by finding a stress function analogous to the ordinary torsion function and solving the problem in toroidal harmonics.

In this paper an approximate solution is obtained using the principle of least work. A stress function is found satisfying the equations of equilibrium and the boundary conditions and whose corresponding stresses make the strain energy a minimum. The solution of the differential equation of compatibility has therefore been replaced by the problem of minimizing the strain energy. In the energy method, the condition of minimum strain energy is equivalent to satisfying compatibility not in a point by point sense, but "on the average" throughout the body.

The purpose of this investigation has been to answer two questions in the author's mind. Namely, in view of the fact that nowhere was the author able to find the energy method used in the literature:

(1) Can the problem be solved by this method, and how do the results compare with those of other solutions?

(2) Does the problem particularly lend itself to solution by energy methods?

It was found that the problem is not adaptable to an exact solution by energy methods, but by making some approximations, excellent results are obtained that agree very closely with Frieberger's exact solution.

FORMULATION OF THE PROBLEM

We will consider a sector of a circular ring with mean radius of curvature $\underline{R}$ and cross-sectional radius $\underline{a}$. A load $\underline{P}$ is applied to one terminal cross-section as shown in Fig. 2, the other remaining fixed. Cylindrical coordinates are used, where the $\underline{z}$ axis coincides with the toroidal axis, and the axis of the ring sector lies in the $\underline{r\theta}$ plane.

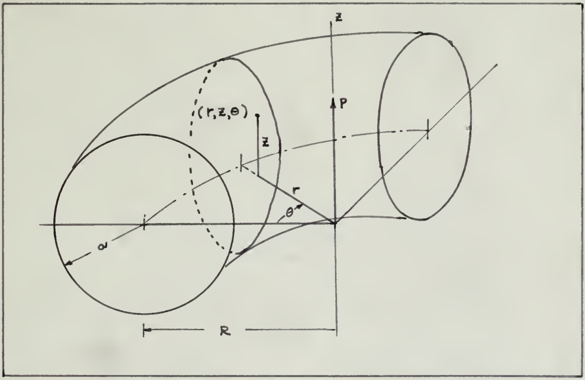

Fig. 2.

$\underline{\theta}$ increases positively as shown in the figure and $\underline{r}$ increases outward from the toroidal axis. Later in the solution the coordinates will be transformed, but for the present purpose of establishing a stress function satisfying the equations of equilibrium, cylindrical coordinates are most convenient.

Assuming zero body forces, from THEORY OF ELASTICITY, Timoshenko & Goodier, Equations (170) the differential equations of equilibrium are

FORMULATION OF THE PROBLEM

We will consider a sector of a circular ring with mean radius of curvature R and cross-sectional radius a. A load P is applied to one terminal cross-section as shown in Fig. 2, the other remaining fixed. Cylindrical coordinates are used, where the z axis coincides with the terminal axis, and the axis of the ring sector lies in the xy plane.

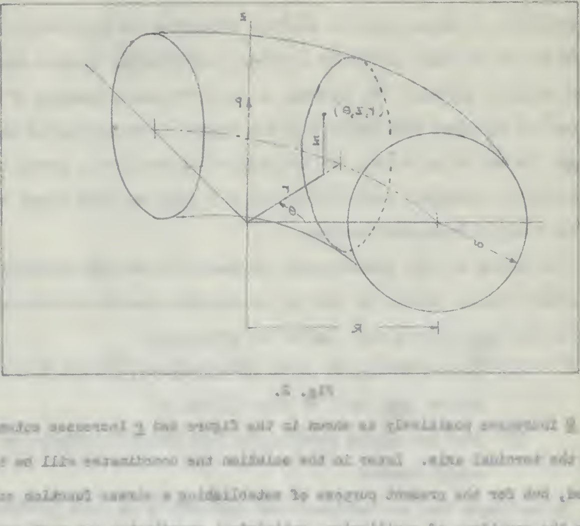

Fig. 2.

θ increases positively as shown in the figure and r increases outward from the terminal axis. Later in the solution the coordinates will be transformed, but for the present purpose of establishing a stress function satisfying the equations of equilibrium, cylindrical coordinates are most convenient.

Assuming zero body forces, from THEORY OF ELASTICITY, Timoshenko & Goodier, Equations (170) the differential equations of equilibrium are

$$(1) \text{——} \begin{cases} \dfrac{\partial \sigma_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{r\theta}}{\partial \theta} + \dfrac{\partial \tau_{rz}}{\partial z} + \dfrac{\sigma_r - \sigma_\theta}{r} = 0 \\ \dfrac{\partial \tau_{rz}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{z\theta}}{\partial \theta} + \dfrac{\partial \sigma_z}{\partial z} + \dfrac{\tau_{rz}}{r} = 0 \\ \dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_\theta}{\partial \theta} + \dfrac{\partial \tau_{z\theta}}{\partial z} + \dfrac{2\tau_{r\theta}}{r} = 0 \end{cases}$$

Using the same assumptions made by Göhner in this case, namely that the only non-vanishing stresses are $\tau_{r\theta}$, $\tau_{z\theta}$ and that the stress distribution in any cross-section is independent of θ these reduce to

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{z\theta}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0$$

This may also be written

$$\left[\frac{\partial}{\partial r}\left(r^2 \tau_{r\theta}\right) + \frac{\partial}{\partial z}\left(r^2 \tau_{z\theta}\right)\right] = 0$$

A stress function ϕ satisfying the above is

$$GR^2 \frac{\partial \phi}{\partial z} = r^2 \tau_{r\theta} \qquad GR^2 \frac{\partial \phi}{\partial r} = -r^2 \tau_{z\theta}$$

Where G is a constant (actually the modulus of rigidity).

Therefore the stresses may be expressed as

$$(2) \text{——} \quad \tau_{r\theta} = \frac{GR^2}{r^2}\frac{\partial \phi}{\partial z} \quad \text{and} \quad \tau_{z\theta} = -\frac{GR^2}{r^2}\frac{\partial \phi}{\partial r}$$

At this point it is convenient to transform the cylindrical coordinates r z θ into toroidal coordinates ρ, ψ, θ (refer to Fig. 3).

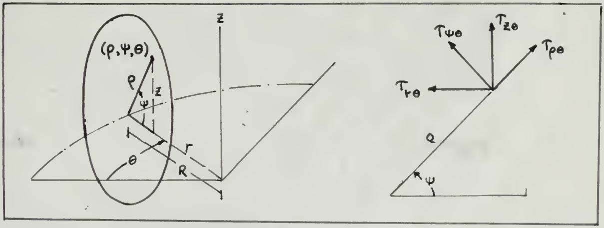

Fig. 3.

If ϕ is a function of r and z, where $\begin{array}{l} r = R - \rho\cos\psi \\ r = \rho\sin\psi \end{array}$ from Fig. 3.

Then

$$\frac{\partial\phi}{\partial\rho} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial\rho} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial\rho} \qquad \frac{\partial r}{\partial\rho} = -\cos\psi \qquad \frac{\partial r}{\partial\psi} = \rho\sin\psi$$

$$\frac{\partial\phi}{\partial\psi} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial\psi} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial\psi} \quad \text{where} \quad \frac{\partial z}{\partial\rho} = \sin\psi \qquad \frac{\partial z}{\partial\psi} = \rho\cos\psi$$

Substituting

$$\frac{\partial\phi}{\partial\rho} = \frac{\partial\phi}{\partial r}(-\cos\psi) + \frac{\partial\phi}{\partial z}(\sin\psi)$$

$$\frac{\partial\phi}{\partial\psi} = \frac{\partial\phi}{\partial r}(\rho\sin\psi) + \frac{\partial\phi}{\partial z}(\rho\cos\psi)$$

Solving for $\frac{\partial\phi}{\partial r}$ and $\frac{\partial\phi}{\partial z}$

$$(3) \text{——} \quad \left\{ \begin{array}{l} \dfrac{\partial\phi}{\partial r} = \left(\dfrac{\sin\psi}{\rho}\right)\dfrac{\partial\phi}{\partial\psi} - (\cos\psi)\dfrac{\partial\phi}{\partial\rho} \\[2ex] \dfrac{\partial\phi}{\partial z} = \left(\dfrac{\cos\psi}{\rho}\right)\dfrac{\partial\phi}{\partial\psi} + (\sin\psi)\dfrac{\partial\phi}{\partial\rho} \end{array} \right.$$

In a plane cross-section determined by θ a constant

$$(4) \text{——} \quad \left\{ \begin{array}{l} \tau_{\rho\theta} = -\tau_{r\theta}\cos\psi + \tau_{z\theta}\sin\psi \\[2ex] \tau_{\psi\theta} = \tau_{r\theta}\sin\psi + \tau_{z\theta}\cos\psi \end{array} \right.$$

Using Equations (2), (3) and (4) the following result is obtained.

$$\tau_{\rho\theta} = -\frac{GR^2}{(R-\rho\cos\psi)^2}\left[\frac{\cos\psi}{\rho}\frac{\partial\phi}{\partial\psi} + \sin\psi\frac{\partial\phi}{\partial\rho}\right] - \frac{GR^2\sin\psi}{(R-\rho\cos\psi)^2}\left[\frac{\sin\psi}{\rho}\frac{\partial\phi}{\partial\psi} - \cos\psi\frac{\partial\phi}{\partial\rho}\right]$$

$$\tau_{\psi\theta} = \frac{GR^2}{(R-\rho\cos\psi)^2}\left[\frac{\cos\psi}{\rho}\frac{\partial\phi}{\partial\psi} + \sin\psi\frac{\partial\phi}{\partial\rho}\right] - \frac{GR^2\cos\psi}{(R-\rho\cos\psi)^2}\left[\frac{\sin\psi}{\rho}\frac{\partial\phi}{\partial\psi} - \cos\psi\frac{\partial\phi}{\partial\rho}\right]$$

Reducing

$$(5) \text{——} \quad \tau_{\rho\theta} = -\frac{GR^2}{(R-\rho\cos\psi)^2}\frac{1}{\rho}\frac{\partial\phi}{\partial\psi} \quad \text{and} \quad \tau_{\psi\theta} = \frac{GR^2}{(R-\rho\cos\psi)^2}\frac{\partial\phi}{\partial\rho}$$

If ϕ is a function of r and z, where $r = R - \rho\cos\psi$, $z = \rho\sin\psi$ (from Fig. 3).

Thus

$$\frac{\partial\phi}{\partial\rho} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial\rho} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial\rho} \qquad \text{where} \qquad \frac{\partial r}{\partial\rho} = -\cos\psi \qquad \frac{\partial r}{\partial\psi} = \rho\sin\psi$$

$$\frac{\partial\phi}{\partial\psi} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial\psi} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial\psi} \qquad\qquad \frac{\partial z}{\partial\rho} = \sin\psi \qquad \frac{\partial z}{\partial\psi} = \rho\cos\psi$$

Substituting

$$\frac{\partial\phi}{\partial\rho} = \frac{\partial\phi}{\partial r}(-\cos\psi) + \frac{\partial\phi}{\partial z}(\sin\psi)$$

$$\frac{\partial\phi}{\partial\psi} = \frac{\partial\phi}{\partial r}(\rho\sin\psi) + \frac{\partial\phi}{\partial z}(\rho\cos\psi)$$

Solving for $\frac{\partial\phi}{\partial r}$ and $\frac{\partial\phi}{\partial z}$

$$(3)\quad \left.\begin{aligned} \frac{\partial\phi}{\partial r} &= \left(\frac{\sin\psi}{\rho}\right)\frac{\partial\phi}{\partial\psi} - (\cos\psi)\frac{\partial\phi}{\partial\rho} \\ \frac{\partial\phi}{\partial z} &= \left(\frac{\cos\psi}{\rho}\right)\frac{\partial\phi}{\partial\psi} + (\sin\psi)\frac{\partial\phi}{\partial\rho} \end{aligned}\right\}$$

In a plane cross-section determined by θ a constant

$$(4)\quad \left.\begin{aligned} \tau_{\rho\theta} &= -\tau_{r\theta}\cos\psi + \tau_{z\theta}\sin\psi \\ \tau_{\psi\theta} &= \tau_{r\theta}\sin\psi + \tau_{z\theta}\cos\psi \end{aligned}\right\}$$

Using equations (2), (3) and (4) the following result is obtained:

$$\tau_{\rho\theta} = -\frac{GR^2}{(R-\rho\cos\psi)^2}\left[\frac{\cos\psi}{\rho}\frac{\partial\phi}{\partial\psi} + \sin\psi\frac{\partial\phi}{\partial\rho}\right] - \frac{GR^2\sin\psi}{(R-\rho\cos\psi)^2}\left[\frac{\sin\psi}{\rho}\frac{\partial\phi}{\partial\psi} - \cos\psi\frac{\partial\phi}{\partial\rho}\right]$$

$$\tau_{\psi\theta} = \frac{GR^2}{(R-\rho\cos\psi)^2}\left[\frac{\cos\psi}{\rho}\frac{\partial\phi}{\partial\psi} + \sin\psi\frac{\partial\phi}{\partial\rho}\right] - \frac{GR^2\cos\psi}{(R-\rho\cos\psi)^2}\left[\frac{\sin\psi}{\rho}\frac{\partial\phi}{\partial\psi} - \cos\psi\frac{\partial\phi}{\partial\rho}\right]$$

Reducing

$$(5)\quad \tau_{\rho\theta} = -\frac{GR^2}{(R-\rho\cos\psi)^2}\frac{1}{\rho}\frac{\partial\phi}{\partial\psi} \qquad \text{and} \qquad \tau_{\psi\theta} = \frac{GR^2}{(R-\rho\cos\psi)^2}\frac{\partial\phi}{\partial\rho}$$

The latter expressions relate the stress function and the stresses in the new system of coordinates.

It follows that since the shear stress $\tau_{\rho\theta}$ is normal to the boundary, it must vanish everywhere on the boundary. This is true because the surface of the body is free from any external forces. Using this condition with Equation (5), it is apparent that $\frac{\partial\phi}{\partial\psi}=0$ and ϕ must be constant on the boundary.

The circular ring sector we are considering is a singly connected body, hence the constant may be chosen arbitrarily. Therefore the boundary condition is taken as $\phi=0$ everywhere on the boundary.

The only action on a cross-section is a force P directed along the toroidal axis. This may be resolved into a force and a couple as shown in Fig. 4.

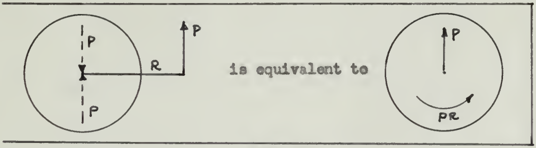

Fig. 4.

It is now seen that the two conditions of static equilibrium to be satisfied are that the resultant stress on a cross-section produce a force P directed along the z axis and a moment about the center PR. These requirements may be written

$$(6)\quad \begin{cases} P=\displaystyle\int_0^a\int_0^{2\pi}\left(\tau_{\rho\theta}\sin\psi+\tau_{\psi\theta}\cos\psi\right)\rho\,d\rho\,d\psi \\ PR=\displaystyle\int_0^a\int_0^{2\pi}\tau_{\psi\theta}\,\rho^2\,d\rho\,d\psi \end{cases}$$

The latter expressions relate the stress function and the stresses in the new system of coordinates.

It follows that since the shear stress $\tau_{\rho\theta}$ is normal to the boundary, it must vanish everywhere on the boundary. This is true because the surface of the body is free from any external forces. Using this condition with Equation (5), it is apparent that $\frac{\partial \phi}{\partial \psi} = 0$ and ϕ must be constant on the boundary.

The circular ring section we are considering is a singly connected body, hence the constant may be chosen arbitrarily. Therefore the boundary condition is taken as $\phi = 0$ everywhere on the boundary.

The only action on a cross-section is a force P directed along the toroidal axis. This may be resolved into a force and a couple as shown in Fig. 4.

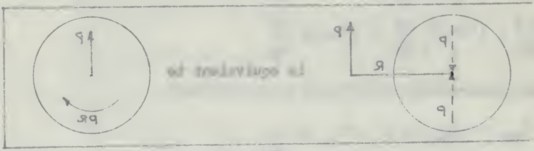

Fig. 4.

It is now seen that the two conditions of static equilibrium to be satisfied are that the resultant stress on a cross-section produce a force P directed along the z axis and a moment about the center PR. These requirements may be written

$$
\left.
\begin{aligned}
P &= \int_0^a \int_0^{2\pi} \left(\tau_{\rho\theta} \sin\psi + \tau_{\psi\theta} \cos\psi \right) \rho \, d\rho \, d\psi \\
PR &= \int_0^a \int_0^{2\pi} \tau_{\psi\theta} \, \rho^2 \, d\rho \, d\psi
\end{aligned}
\right\} \text{———} (6)
$$

The strain energy per unit angle $\underline{\theta}$ is

$$(7)\text{——} \quad U = \frac{1}{2G} \int_0^{a} \int_0^{2\pi} \left(T_{\rho\theta}^2 + T_{\psi\theta}^2 \right) \left(R - \rho \cos\psi \right) \rho \, d\rho \, d\psi$$

The method of solution will now be to take the stress function in the form $\phi = \sum_{i=0}^{N} \alpha_i \phi_i$, where ϕ_i are suitably selected functions of ρ and ψ , each of which satisfies the boundary condition $\phi_i = 0$ when $\rho = a$. The coefficients α_i are constants which are evaluated from the minimum condition of strain energy.

For a first approximation we shall take a function ϕ, satisfying the boundary condition that it vanish everywhere on the boundary, in the form $\phi = (\rho^2-a^2)\left(\alpha_0 + \frac{\alpha_1\rho}{R}\cos\psi\right)$. The reasons for this particular choice are discussed in Appendix A. Taking the partial derivatives of ϕ with respect to the two variables ρ and ψ

$$\frac{\partial\phi}{\partial\rho} = 2\rho\alpha_0 + \frac{\alpha_1(3\rho^2-a^2)}{R}\cos\psi \quad \text{and} \quad \frac{\partial\phi}{\partial\psi} = -\frac{\alpha_1(\rho^2-a^2)}{R}\sin\psi$$

Substituting in Equations (5), the following expressions are obtained for $T_{\rho\theta}$ and $T_{\psi\theta}$.

$$(8) \text{------} T_{\rho\theta} = \frac{GR^2}{(R-\rho\cos\psi)^2}\frac{\alpha_1(\rho^2-a^2)}{R}\sin\psi \quad \text{and} \quad T_{\psi\theta} = \frac{GR^2}{(R-\rho\cos\psi)^2}\left[2\rho\alpha_0 + \frac{\alpha_1(3\rho^2-a^2)}{R}\cos\psi\right]$$

The appearance of the term $\frac{1}{(R-\rho\cos\psi)^2}$ in the stress equations makes the integration required in (6) and (7) very complicated and the results largely unmanageable in the evaluation of the unknown coefficients in ϕ. (See Appendix B). This is particularly true when additional terms are used in ϕ for a higher order of approximation, and in the evaluation of the strain energy where the stresses appear as squared terms.

Since $\frac{\rho}{R}$ is always less than unity, we may write

$$\frac{R^2}{(R-\rho\cos\psi)^2} = \frac{1}{\left(1-\frac{\rho}{R}\cos\psi\right)^2} = 1 + 2\left(\frac{\rho}{R}\right)\cos\psi + 3\left(\frac{\rho}{R}\right)^2\cos^2\psi + \cdots$$

Utilizing this expansion, the exact stress expressions (5) may be approximated as follows

$$T_{\rho\theta} = -\frac{G}{\rho}\left[\left(1 + 2\frac{\rho}{R}\cos\psi\right)\frac{\partial\phi_0}{\partial\psi}\alpha_0 + \frac{\partial\phi_1}{\partial\psi}\alpha_1\right]$$

$$T_{\psi\theta} = G\left[\left(1 + 2\frac{\rho}{R}\cos\psi\right)\frac{\partial\phi_0}{\partial\rho}\alpha_0 + \frac{\partial\phi_1}{\partial\rho}\alpha_1\right]$$

This particular form of approximation accomplishes the desired result

First Approximation

For a first approximation we shall take a function Φ, satisfying the boundary condition that it vanish everywhere on the boundary, in the form $\Phi = (\rho^2 - a^2)\left(\alpha_0 + \frac{\alpha_1 \rho}{R}\cos\psi\right)$. The reasons for this particular choice are discussed in Appendix A. Taking the partial derivatives of Φ with respect to the two variables ρ and ψ

$$\frac{\partial\Phi}{\partial\rho} = 2\rho\alpha_0 + \frac{\alpha_1(3\rho^2 - a^2)}{R}\cos\psi \quad \text{and} \quad \frac{\partial\Phi}{\partial\psi} = -\frac{\alpha_1(\rho^2 - a^2)}{R}\sin\psi$$

Substituting in Equations (5), the following expressions are obtained for $T_{\rho\theta}$ and $T_{\psi\theta}$.

$$\text{(8)} \;—\; T_{\rho\theta} = \frac{GR^2}{(R-\rho\cos\psi)^2}\,\frac{\alpha_1(\rho^2 - a^2)}{R}\sin\psi \quad \text{and} \quad T_{\psi\theta} = \frac{GR^2}{(R-\rho\cos\psi)^2}\left[2\rho\alpha_0 + \frac{\alpha_1(3\rho^2 - a^2)}{R}\cos\psi\right]$$

The appearance of the term $\frac{1}{(R-\rho\cos\psi)^2}$ in the stress equations makes the integration required in (6) and (7) very complicated and the results largely unmanageable in the evaluation of the unknown coefficients in Φ. (See Appendix B). This is particularly true when additional terms are used in Φ for a higher order of approximation, and in the evaluation of the strain energy where the stresses appear as squared terms.

Since $\frac{\rho}{R}$ is always less than unity, we may write

$$\frac{R^2}{(R-\rho\cos\psi)^2} = \frac{1}{\left(1 - \frac{\rho}{R}\cos\psi\right)^2} = 1 + 2\left(\frac{\rho}{R}\right)\cos\psi + 3\left(\frac{\rho}{R}\right)^2\cos^2\psi + \cdots$$

Utilizing this expansion, the exact stress expressions (5) may be approximated as follows

$$T_{\rho\theta} = -\frac{G}{\rho}\left[\left(1 + 2\frac{\rho}{R}\cos\psi\right)\frac{\partial\Phi_0}{\partial\psi}\alpha_0 + \frac{\partial\Phi_1}{\partial\psi}\alpha_1\right]$$

$$T_{\psi\theta} = G\left[\left(1 + 2\frac{\rho}{R}\cos\psi\right)\frac{\partial\Phi_0}{\partial\rho}\alpha_0 + \frac{\partial\Phi_1}{\partial\rho}\alpha_1\right]$$

This particular form of approximation accomplishes the desired result

of limiting the highest power to which the ratios $\frac{a}{R}$ and $\frac{\rho}{R}$ appear in the stress equations.

Since $\quad \phi_0 = (\rho^2 - a^2) \quad$ and $\quad \phi_1 = \frac{\rho(\rho^2 - a^2)}{R}\cos\psi$

The partial derivatives are

$$\frac{\partial\phi_0}{\partial\rho} = 2\rho \qquad \frac{\partial\phi_1}{\partial\rho} = \frac{(3\rho^2 - a^2)}{R}\cos\psi$$

$$\frac{\partial\phi_0}{\partial\psi} = 0 \qquad \frac{\partial\phi_1}{\partial\psi} = -\frac{\rho(\rho^2 - a^2)}{R}\sin\psi$$

Substituting, we arrive at the following approximate expressions for the stresses.

$$(9)\text{----------}\begin{cases} T_{\rho\theta} = G\left[\frac{\alpha_1(\rho^2 - a^2)}{R}\sin\psi\right] \\ T_{\psi\theta} = G\left[2\rho\alpha_0 + \frac{(4\rho^2\alpha_0 + 3\rho^2\alpha_1 - a^2\alpha_1)}{R}\cos\psi\right] \end{cases}$$

Substituting these values of $T_{\rho\theta}$ and $T_{\psi\theta}$ in the first of Equations (6), and integrating we obtain

$$P = \frac{G\pi a^4}{R}\alpha_0 \qquad \therefore \quad \alpha_0 = \frac{PR}{G\pi a^4}$$

The same result is obtained from the second condition of Equations (6).

It follows that α_0 is fixed by the requirements of static equilibrium and α_1 may now be determined by the condition of minimum strain energy that $\frac{\partial U}{\partial\alpha_1} = 0$.

From Equation (7)

$$\frac{\partial U}{\partial\alpha_1} = \frac{1}{G}\int_0^a\int_0^{2\pi}\left(T_{\rho\theta}\frac{\partial T_{\rho\theta}}{\partial\alpha_1} + T_{\psi\theta}\frac{\partial T_{\psi\theta}}{\partial\alpha_1}\right)(R - \rho\cos\psi)\rho\, d\rho\, d\psi$$

Substituting the stresses from Equation (9) and integrating

$$\frac{\partial U}{\partial\alpha_1} = \frac{R\pi}{G}\left[\left(\frac{1}{2}\frac{a^6}{R^2}\right)\alpha_0 + \left(\frac{2}{3}\frac{a^6}{R^2}\right)\alpha_1\right]$$

Setting $\frac{\partial U}{\partial\alpha_1} = 0$ and solving for α_1

of limiting the highest power to which the ratios $\frac{a}{R}$ and $\frac{\rho}{R}$ appear in the stress equations.

Thus $\Phi_0 = (\rho^2 - a^2)$ and $\Phi_1 = \frac{\rho(\rho^2 - a^2)}{R}\cos\psi$

The partial derivatives are

$$\frac{\partial \Phi_0}{\partial \rho} = 2\rho \qquad \frac{\partial \Phi_1}{\partial \rho} = \frac{(3\rho^2 - a^2)}{R}\cos\psi$$

$$\frac{\partial \Phi_0}{\partial \psi} = 0 \qquad \frac{\partial \Phi_1}{\partial \psi} = -\frac{\rho(\rho^2 - a^2)}{R}\sin\psi$$

Substituting, we arrive at the following approximate expressions for the stresses.

$$\left.\begin{aligned} T_{\rho\theta} &= G\left[\frac{\alpha_1(\rho^2 - a^2)}{R}\sin\psi\right] \\ T_{\psi\theta} &= G\left[2\rho\alpha_0 + \frac{(4\rho^2\alpha_0 + 3\rho^2\alpha_1 - a^2\alpha_1)}{R}\cos\psi\right] \end{aligned}\right\} \text{------------} (9)$$

Substituting these values of $T_{\rho\theta}$ and $T_{\psi\theta}$ in the first of Equations (6), and integrating we obtain

$$P = \frac{G\pi a^4}{R}\alpha_0 \qquad \therefore \quad \alpha_0 = \frac{PR}{G\pi a^4}$$

The same result is obtained from the second condition of Equations (6). It follows that α_0 is fixed by the requirements of static equilibrium and α_1 may now be determined by the condition of minimum strain energy that $\frac{\partial U}{\partial \alpha_1} = 0$.

From Equation (7)

$$\frac{\partial U}{\partial \alpha_1} = \frac{1}{G}\int_0^a\int_0^{2\pi}\left(T_{\rho\theta}\frac{\partial T_{\rho\theta}}{\partial \alpha_1} + T_{\psi\theta}\frac{\partial T_{\psi\theta}}{\partial \alpha_1}\right)(R - \rho\cos\psi)\rho\, d\rho\, d\psi$$

Substituting the stresses from Equation (9) and integrating

$$\frac{\partial U}{\partial \alpha_1} = \frac{R\pi}{G}\left[\left(\frac{1}{2}\frac{a^6}{R^2}\right)\alpha_0 + \left(\frac{2}{3}\frac{a^6}{R^2}\right)\alpha_1\right]$$

Setting $\frac{\partial U}{\partial \alpha_1} = 0$ and solving for α_1

Using these results in Equations (9) we arrive at the expressions for the first approximation of the stress distribution in a cross-section of the incomplete tore

$$
(10)\qquad
\begin{cases}
\tau_{\rho\theta} = -\dfrac{PR}{\pi a^4}\left[\dfrac{3}{4}\,\dfrac{(\rho^2-a^2)}{R}\sin\psi\right] \\[2ex]
\tau_{\psi\theta} = \dfrac{PR}{\pi a^4}\left[2\rho + \dfrac{(7\rho^2+3a^2)}{R}\cos\psi\right]
\end{cases}
$$

At the point of maximum stress where $\rho = a$ and $\psi = 0$ the above reduce to

$$
(11)\qquad
\begin{cases}
\tau_{\rho\theta} = 0 \\[2ex]
\left[\tau_{\psi\theta}\right]_{max} = \dfrac{2PR}{\pi a^3}\left[1 + \dfrac{5}{4}\left(\dfrac{a}{R}\right)\right]
\end{cases}
$$

It is interesting to note at this point that for this particular solution, one of the unknown coefficients in ϕ is determined directly from the requirements of static equilibrium, and the other directly from the minimum strain energy condition without constraint arising from static equilibrium.

SECOND APPROXIMATION

A closer approximation to the true stress conditions will result if higher order terms of a suitable nature are used in the stress function. We shall now take ϕ as

$$\phi = (\rho^2-a^2)\left(\alpha_0 + \frac{\alpha_1\rho}{R}\cos\psi + \frac{\alpha_2\rho^2}{R^2}\cos^2\psi + \frac{\alpha_3\rho^2}{R^2}\sin^2\psi + \frac{\alpha_4 a^2}{R^2}\right)$$

Reasons for this particular choice of functions are discussed in Appendix A.

Again employing the binomial expansion of $\frac{1}{(R-\rho\cos\psi)^2}$ we write approximate expressions for $\tau_{\rho\theta}$ and $\tau_{\psi\theta}$.

$$\tau_{\rho\theta} = -\frac{G}{\rho}\left[\left(1+2\frac{\rho}{R}\cos\psi+3\frac{\rho^2}{R^2}\cos^2\psi\right)\frac{\partial\phi_0}{\partial\psi}\alpha_0 + \left(1+2\frac{\rho}{R}\cos\psi\right)\frac{\partial\phi_1}{\partial\psi}\alpha_1 + \frac{\partial\phi_2}{\partial\psi}\alpha_2 + \frac{\partial\phi_3}{\partial\psi}\alpha_3 + \frac{\partial\phi_4}{\partial\psi}\alpha_4\right]$$

$$\tau_{\psi\theta} = G\left[\left(1+2\frac{\rho}{R}\cos\psi+3\frac{\rho^2}{R^2}\cos^2\psi\right)\frac{\partial\phi_0}{\partial\rho}\alpha_0 + \left(1+2\frac{\rho}{R}\cos\psi\right)\frac{\partial\phi_1}{\partial\rho}\alpha_1 + \frac{\partial\phi_2}{\partial\rho}\alpha_2 + \frac{\partial\phi_3}{\partial\rho}\alpha_3 + \frac{\partial\phi_4}{\partial\rho}\alpha_4\right]$$

Where

$$\phi_0 = (\rho^2-a^2) \qquad \phi_2 = \frac{\rho^2(\rho^2-a^2)}{R^2}\cos^2\psi \qquad \phi_4 = \frac{a^2(\rho^2-a^2)}{R^2}$$

$$\phi_1 = \frac{\rho(\rho^2-a^2)}{R}\cos\psi \qquad \phi_3 = \frac{\rho^2(\rho^2-a^2)}{R^2}\sin^2\psi$$

This is an extension of the device used before to limit the highest power to which the ratios $\frac{a}{R}$ and $\frac{\rho}{R}$ appear in each term of the stress equations. Since $\frac{a}{R}$ and $\frac{\rho}{R}$ occur in a like manner in ϕ_2, ϕ_3 and ϕ_4, these latter terms are grouped together and treated in similar fashion when introduced into the approximate expressions for the stresses.

Taking the partial derivatives, substituting and rearranging the terms for convenient integration, the following approximate expressions for $\tau_{\rho\theta}$ and $\tau_{\psi\theta}$ are obtained.

$$(12)\text{------}\left\{\begin{aligned}\tau_{\rho\theta} &= G\left[\frac{\alpha_1(\rho^2-a^2)}{R}\sin\psi + \frac{2\rho(\alpha_1+\alpha_2-\alpha_3)(\rho^2-a^2)}{R^2}\sin\psi\cos\psi\right]\\ \tau_{\psi\theta} &= G\left\{\left[\frac{6\rho^3\alpha_0}{R^2} + \frac{2\rho\alpha_1(3\rho^2-a^2)}{R^2} + \frac{2\rho\alpha_2(2\rho^2-a^2)}{R^2}\right]\cos^2\psi + \left[\frac{4\rho^2\alpha_0}{R} + \frac{\alpha_1(3\rho^2-a^2)}{R}\right]\cos\psi + 2\rho\alpha_0 + \right.\\ &\qquad \left.\frac{2\rho\alpha_3(2\rho^2-a^2)}{R^2}\sin^2\psi + \frac{2a^2\alpha_4\rho}{R^2}\right\}\end{aligned}\right.$$

SECOND APPROXIMATION

A closer approximation to the true stress conditions will result if higher order terms of a suitable nature are used in the stress function. We shall now take Φ as

$$\Phi = (\rho^2 - a^2)\left(\alpha_0 + \frac{\alpha_1 \rho}{R}\cos\psi + \frac{\alpha_2 \rho^2}{R^2}\cos^2\psi + \frac{\alpha_3 \rho^2}{R^2}\sin^2\psi + \frac{\alpha_4 a^2}{R^2}\right)$$

Reasons for this particular choice of functions are discussed in Appendix A.

Again employing the binomial expansion of $\frac{1}{(R-\rho\cos\psi)}$, we write approximate expressions for $\tau_{\rho\theta}$ and $\tau_{\psi\theta}$.

$$\tau_{\rho\theta} = -\frac{G}{\rho}\left[\left(1 + 2\frac{\rho}{R}\cos\psi + 3\frac{\rho^2}{R^2}\cos^2\psi\right)\frac{\partial\Phi_0}{\partial\psi}\alpha_0 + \left(1 + 2\frac{\rho}{R}\cos\psi\right)\frac{\partial\Phi_1}{\partial\psi}\alpha_1 + \frac{\partial\Phi_2}{\partial\psi}\alpha_2 + \frac{\partial\Phi_3}{\partial\psi}\alpha_3 + \frac{\partial\Phi_4}{\partial\psi}\alpha_4\right]$$

$$\tau_{\psi\theta} = G\left[\left(1 + 2\frac{\rho}{R}\cos\psi + 3\frac{\rho^2}{R^2}\cos^2\psi\right)\frac{\partial\Phi_0}{\partial\rho}\alpha_0 + \left(1 + 2\frac{\rho}{R}\cos\psi\right)\frac{\partial\Phi_1}{\partial\rho}\alpha_1 + \frac{\partial\Phi_2}{\partial\rho}\alpha_2 + \frac{\partial\Phi_3}{\partial\rho}\alpha_3 + \frac{\partial\Phi_4}{\partial\rho}\alpha_4\right]$$

where

$$\Phi_0 = (\rho^2 - a^2) \qquad \Phi_2 = \frac{\rho^2(\rho^2-a^2)}{R^2}\cos^2\psi \qquad \Phi_4 = \frac{a^2(\rho^2-a^2)}{R^2}$$

$$\Phi_1 = \frac{\rho(\rho^2-a^2)}{R}\cos\psi \qquad \Phi_3 = \frac{\rho^2(\rho^2-a^2)}{R^2}\sin^2\psi$$

This is an extension of the device used before to limit the highest power to which the ratios $\frac{a}{R}$ and $\frac{\rho}{R}$ appear in each term of the stress equations. Since $\frac{a}{R}$ and $\frac{\rho}{R}$ occur in a like manner in Φ_2, Φ_3 and Φ_4, these latter terms are grouped together and treated in similar fashion when introduced into the approximate expressions for the stresses.

Taking the partial derivatives, substituting and rearranging the terms for convenient integration, the following approximate expressions for $\tau_{\rho\theta}$ and $\tau_{\psi\theta}$ are obtained.

$$\left.\begin{aligned}
\tau_{\rho\theta} &= G\left[\frac{\alpha_1(\rho^2-a^2)}{R}\sin\psi + \frac{2\rho(\alpha_1+\alpha_2-\alpha_3)(\rho^2-a^2)}{R^2}\sin\psi\cos\psi\right] \\
\tau_{\psi\theta} &= G\left\{\left[\frac{6\rho^3\alpha_2}{R^2} + \frac{2\rho\alpha_1(3\rho^2-a^2)}{R^2} + \frac{2\rho\alpha_2(2\rho^2-a^2)}{R^2}\right]\cos^2\psi + \left[\frac{4\rho^2\alpha_0}{R} + \frac{\alpha_1(3\rho^2-a^2)}{R}\right]\cos\psi + 2\rho\alpha_0 + \right. \\
&\quad \left. \frac{2\rho\alpha_3(2\rho^2-a^2)}{R^2}\sin^2\psi + \frac{2a^2\alpha_4\rho}{R^2}\right\}
\end{aligned}\right\} \text{——(12)}$$

From the first of the static equilibrium conditions in (6) (that the resultant stress must produce a force $\underline{P}$ in the $\underline{Z}$ direction) it is again found that $\alpha_0 = \frac{PR}{G\pi a^4}$.

The second static equilibrium condition (that the resultant stress must produce a moment about the center equal to $\underline{PR}$) gives the following result.

$$\frac{PR}{G\pi a^4} = \left[\left(1 + \frac{a^2}{R^2}\right)\alpha_0 + \left(\frac{1}{2}\frac{a^2}{R^2}\right)\alpha_1 + \left(\frac{1}{6}\frac{a^2}{R^2}\right)\alpha_2 + \left(\frac{1}{6}\frac{a^2}{R^2}\right)\alpha_3 + \left(\frac{a^2}{R^2}\right)\alpha_4\right]$$

However, since $\alpha_0 = \frac{PR}{G\pi a^4}$

$$\left(\frac{1}{2}\frac{a^2}{R^2}\right)\alpha_1 + \left(\frac{1}{6}\frac{a^2}{R^2}\right)\alpha_2 + \left(\frac{1}{6}\frac{a^2}{R^2}\right)\alpha_3 + \left(\frac{a^2}{R^2}\right)\alpha_4 + \left(\frac{a^2}{R^2}\right)\left(\frac{PR}{G\pi a^4}\right) = 0$$

Since α_1 , α_2 , α_3 and α_4 will ultimately all contain the factor $\frac{PR}{G\pi a^4}$ some simplification of the algebra will be afforded if we make the following substitutions

$$\beta_n = \alpha_n\left(\frac{G\pi a^4}{PR}\right) \qquad \text{where} \quad n = 1, 2, 3, 4$$

Finally the constraining function derived from the conditions of static equilibrium to be used in minimizing the strain energy is

(13)------------ $$\left(\frac{1}{2}\frac{a^2}{R^2}\right)\beta_1 + \left(\frac{1}{6}\frac{a^2}{R^2}\right)\beta_2 + \left(\frac{1}{6}\frac{a^2}{R^2}\right)\beta_3 + \left(\frac{a^2}{R^2}\right)\beta_4 + \frac{a^2}{R^2} = 0$$

The work involved in obtaining the partial derivatives of the strain energy with respect to the unknown coefficients α_1 , α_2 , α_3 and α_4 will be simplified by differentiating under the integral sign.

Therefore

$$\frac{\partial U}{\partial \alpha_n} = \frac{R}{G}\int_0^a\int_0^{2\pi}\left(T_{\rho\theta}\frac{\partial T_{\rho\theta}}{\partial \alpha_n} + T_{\psi\theta}\frac{\partial T_{\psi\theta}}{\partial \alpha_n}\right)\left(1 - \frac{\rho}{R}\cos\psi\right)\rho\, d\rho\, d\psi$$

$\frac{\partial U}{\partial \alpha_0}$ is not required since α_0 has already been evaluated from the

conditions of static equilibrium. Since $\frac{\partial U}{\partial \alpha_n} = (\text{Constant}) \frac{\partial U}{\partial \beta_n}$, it is convenient here to take $\frac{\partial U}{\partial \beta_n}$. After substituting for $\tau_{\rho\theta}$ and $\tau_{\psi\theta}$ from Equations (12) and performing the required integration, the following expressions are obtained.

$$
(14)\text{----}\begin{cases}
\frac{\partial U}{\partial \beta_1} = \frac{\pi a^6}{GR}\left[\left(\frac{3}{2} - \frac{5}{16}\frac{a^2}{R^2}\right) + \frac{2}{3}\beta_1 + \frac{13}{48}\frac{a^2}{R^2}\beta_2 + \frac{1}{16}\frac{a^2}{R^2}\beta_3 + \frac{1}{2}\frac{a^2}{R^2}\beta_4\right] \\
\frac{\partial U}{\partial \beta_2} = \frac{\pi a^6}{GR}\left[\left(\frac{1}{3} - \frac{1}{4}\frac{a^2}{R^2}\right) + \frac{13}{16}\frac{a^2}{R^2}\beta_1 + \frac{7}{24}\frac{a^2}{R^2}\beta_2 + \frac{1}{24}\frac{a^2}{R^2}\beta_3 + \frac{1}{3}\frac{a^2}{R^2}\beta_4\right] \\
\frac{\partial U}{\partial \beta_3} = \frac{\pi a^6}{GR}\left[\left(\frac{1}{3} + \frac{1}{12}\frac{a^2}{R^2}\right) + \frac{1}{16}\frac{a^2}{R^2}\beta_1 + \frac{1}{24}\frac{a^2}{R^2}\beta_2 + \frac{7}{24}\frac{a^2}{R^2}\beta_3 + \frac{1}{3}\frac{a^2}{R^2}\beta_4\right] \\
\frac{\partial U}{\partial \beta_4} = \frac{\pi a^6}{GR}\left[\left(2 + \frac{2}{3}\frac{a^2}{R^2}\right) + \frac{1}{2}\frac{a^2}{R^2}\beta_1 + \frac{1}{3}\frac{a^2}{R^2}\beta_2 + \frac{1}{3}\frac{a^2}{R^2}\beta_3 + 2\frac{a^2}{R^2}\beta_4\right]
\end{cases}
$$

To minimize the strain energy and evaluate the unknown coefficients β_1, β_2, β_3 and β_4, the method of Lagrangian multipliers will be used with the constraining function (13) established by the requirements of static equilibrium. The constant $\frac{\pi a^6}{GR}$ appearing in the partial derivatives of the strain energy will be incorporated in the multiplier. Letting λ be a Lagrangian multiplier, and $f(\beta_1, \beta_2, \beta_3, \beta_4) = 0$ the constraining function, we may write

$$\frac{\partial U}{\partial \beta_n} + \lambda \frac{\partial f}{\partial \beta_n} = 0$$

$$f(\beta_1, \beta_2, \beta_3, \beta_4) = 0 \qquad \text{where } n = 1, 2, 3, 4$$

Using (13) and (14) in the above, and the fact that

$$\frac{\partial f}{\partial \beta_1} = \frac{1}{2}\frac{a^2}{R^2} \qquad \frac{\partial f}{\partial \beta_2} = \frac{1}{6}\frac{a^2}{R^2} \qquad \frac{\partial f}{\partial \beta_3} = \frac{1}{6}\frac{a^2}{R^2} \qquad \frac{\partial f}{\partial \beta_4} = \frac{a^2}{R^2}$$

we arrive at the following set of equations, the solution of which will evaluate β_1, β_2, β_3, β_4 and determine the stress function.

$$
(15)\text{------}\left\{\begin{aligned}
&\left(\frac{3}{2}-\frac{5}{16}\frac{a^2}{R^2}\right)+\frac{2}{3}\beta_1+\frac{13}{48}\frac{a^2}{R^2}\beta_2+\frac{1}{16}\frac{a^2}{R^2}\beta_3+\frac{1}{2}\frac{a^2}{R^2}\beta_4=-\frac{\lambda'}{2}\frac{a^2}{R^2}\\
&\left(\frac{1}{3}-\frac{1}{4}\frac{a^2}{R^2}\right)+\frac{13}{16}\frac{a^2}{R^2}\beta_1+\frac{7}{24}\frac{a^2}{R^2}\beta_2+\frac{1}{24}\frac{a^2}{R^2}\beta_3+\frac{1}{3}\frac{a^2}{R^2}\beta_4=-\frac{\lambda'}{6}\frac{a^2}{R^2}\\
&\left(\frac{1}{3}+\frac{1}{12}\frac{a^2}{R^2}\right)+\frac{1}{16}\frac{a^2}{R^2}\beta_1+\frac{1}{24}\frac{a^2}{R^2}\beta_2+\frac{7}{24}\frac{a^2}{R^2}\beta_3+\frac{1}{3}\frac{a^2}{R^2}\beta_4=-\frac{\lambda'}{6}\frac{a^2}{R^2}\\
&\left(2+\frac{2}{3}\frac{a^2}{R^2}\right)+\frac{1}{2}\frac{a^2}{R^2}\beta_1+\frac{1}{3}\frac{a^2}{R^2}\beta_2+\frac{1}{3}\frac{a^2}{R^2}\beta_3+2\frac{a^2}{R^2}\beta_4=-\lambda'\frac{a^2}{R^2}\\
&\frac{a^2}{R^2}+\frac{1}{2}\frac{a^2}{R^2}\beta_1+\frac{1}{6}\frac{a^2}{R^2}\beta_2+\frac{1}{6}\frac{a^2}{R^2}\beta_3+\frac{a^2}{R^2}\beta_4=0
\end{aligned}\right.
$$

where $\lambda' = \left(\frac{GR}{\pi a^6}\right)\lambda$

Solving for β_1, β_2, β_3 and β_4

$$\beta_1 = -\frac{3}{4}\left[\frac{1-\frac{37}{72}\frac{a^2}{R^2}}{1-\frac{43}{192}\frac{a^2}{R^2}}\right] \qquad \beta_2 = \frac{57}{96}\left[\frac{1-\frac{37}{72}\frac{a^2}{R^2}}{1-\frac{43}{192}\frac{a^2}{R^2}}\right]-\frac{56}{96}$$

$$\beta_3 = \frac{8}{96}-\frac{3}{96}\left[\frac{1-\frac{37}{72}\frac{a^2}{R^2}}{1-\frac{43}{192}\frac{a^2}{R^2}}\right] \qquad \beta_4 = -\frac{88}{96}+\frac{27}{96}\left[\frac{1-\frac{37}{72}\frac{a^2}{R^2}}{1-\frac{43}{192}\frac{a^2}{R^2}}\right]$$

The corresponding values of α are

$$\alpha_1 = -\frac{3}{4}\frac{PR}{G\pi a^4}\left[\frac{1-\frac{37}{72}\frac{a^2}{R^2}}{1-\frac{43}{192}\frac{a^2}{R^2}}\right] \qquad \alpha_2 = \frac{PR}{G\pi a^4}\left\{\frac{57}{96}\left[\frac{1-\frac{37}{72}\frac{a^2}{R^2}}{1-\frac{43}{192}\frac{a^2}{R^2}}\right]-\frac{56}{96}\right\}$$

$$\alpha_3 = \frac{PR}{G\pi a^4}\left\{\frac{8}{96}-\frac{3}{96}\left[\frac{1-\frac{37}{72}\frac{a^2}{R^2}}{1-\frac{43}{192}\frac{a^2}{R^2}}\right]\right\} \qquad \alpha_4 = \frac{PR}{G\pi a^4}\left\{-\frac{88}{96}+\frac{27}{96}\left[\frac{1-\frac{37}{72}\frac{a^2}{R^2}}{1-\frac{43}{192}\frac{a^2}{R^2}}\right]\right\}$$

we arrive at the following set of equations, the solution of which will evaluate $\beta_1, \beta_2, \beta_3, \beta_4$ and determine the stress function.

$$\left.\begin{aligned}
&\left(\frac{3}{2}-\frac{5}{16}\frac{a^2}{R^2}\right)+\frac{2}{3}\beta_1+\frac{13}{48}\frac{a^2}{R^2}\beta_2+\frac{1}{16}\frac{a^2}{R^2}\beta_3+\frac{1}{2}\frac{a^2}{R^2}\beta_4=-\frac{\lambda'}{2}\frac{a^2}{R^2}\\
&\left(\frac{1}{3}-\frac{1}{4}\frac{a^2}{R^2}\right)+\frac{13}{16}\frac{a^2}{R^2}\beta_1+\frac{7}{24}\frac{a^2}{R^2}\beta_2+\frac{1}{24}\frac{a^2}{R^2}\beta_3+\frac{1}{3}\frac{a^2}{R^2}\beta_4=-\frac{\lambda'}{6}\frac{a^2}{R^2}\\
&\left(\frac{1}{3}+\frac{1}{12}\frac{a^2}{R^2}\right)+\frac{1}{16}\frac{a^2}{R^2}\beta_1+\frac{1}{24}\frac{a^2}{R^2}\beta_2+\frac{7}{24}\frac{a^2}{R^2}\beta_3+\frac{1}{3}\frac{a^2}{R^2}\beta_4=-\frac{\lambda'}{6}\frac{a^2}{R^2}\\
&\left(2+\frac{2}{3}\frac{a^2}{R^2}\right)+\frac{1}{2}\frac{a^2}{R^2}\beta_1+\frac{1}{3}\frac{a^2}{R^2}\beta_2+\frac{1}{3}\frac{a^2}{R^2}\beta_3+2\frac{a^2}{R^2}\beta_4=-\lambda'\frac{a^2}{R^2}\\
&\frac{a^2}{R^2}+\frac{1}{2}\frac{a^2}{R^2}\beta_1+\frac{1}{6}\frac{a^2}{R^2}\beta_2+\frac{1}{6}\frac{a^2}{R^2}\beta_3+\frac{a^2}{R^2}\beta_4=0
\end{aligned}\right\} \quad (15)$$

where $\lambda' = \left(\frac{6R}{\pi a^6}\right)\lambda$

Solving for $\beta_1, \beta_2, \beta_3$ and β_4

$$\beta_1 = -\frac{3}{4}\left[\frac{1-\frac{43}{192}\frac{a^2}{R^2}}{1-\frac{37}{72}\frac{a^2}{R^2}}\right] \qquad \beta_2 = \frac{57}{96}\left[\frac{1-\frac{43}{192}\frac{a^2}{R^2}}{1-\frac{37}{72}\frac{a^2}{R^2}}\right]-\frac{56}{96}$$

$$\beta_3 = \frac{8}{96}-\frac{3}{96}\left[\frac{1-\frac{43}{192}\frac{a^2}{R^2}}{1-\frac{37}{72}\frac{a^2}{R^2}}\right] \qquad \beta_4 = -\frac{88}{96}+\frac{27}{96}\left[\frac{1-\frac{43}{192}\frac{a^2}{R^2}}{1-\frac{37}{72}\frac{a^2}{R^2}}\right]$$

The corresponding values of α are

$$\alpha_1 = -\frac{3}{4}\frac{PR}{6\pi a^4}\left[\frac{1-\frac{43}{192}\frac{a^2}{R^2}}{1-\frac{37}{72}\frac{a^2}{R^2}}\right] \qquad \alpha_2 = \frac{PR}{6\pi a^4}\left\{\frac{57}{96}\left[\frac{1-\frac{43}{192}\frac{a^2}{R^2}}{1-\frac{37}{72}\frac{a^2}{R^2}}\right]-\frac{56}{96}\right\}$$

$$\alpha_3 = \frac{PR}{6\pi a^4}\left\{\frac{8}{96}-\frac{3}{96}\left[\frac{1-\frac{43}{192}\frac{a^2}{R^2}}{1-\frac{37}{72}\frac{a^2}{R^2}}\right]\right\} \qquad \alpha_4 = \frac{PR}{6\pi a^4}\left\{-\frac{88}{96}+\frac{27}{96}\left[\frac{1-\frac{43}{192}\frac{a^2}{R^2}}{1-\frac{37}{72}\frac{a^2}{R^2}}\right]\right\}$$

Using these results in Equations (12), we may now write expressions representing a second approximation of the stress distribution in a cross-section of the incomplete tore.

$$(16)\quad \begin{cases} \tau_{\rho\theta} = -\dfrac{PR}{\pi a^4}\left[\dfrac{3}{4}\delta\left(\rho^2-a^2\right)\dfrac{\sin\psi}{R} + \left(\dfrac{2}{3}+\dfrac{1}{8}\delta\right)\dfrac{\sin 2\psi}{R^2}\right] \quad \text{where } \delta = \left[\dfrac{1-\frac{37}{72}\frac{a^2}{R^2}}{1-\frac{43}{192}\frac{a^2}{R^2}}\right] \\[2ex] \tau_{\psi\theta} = \dfrac{PR}{\pi a^4}\left\{\left[\left(\dfrac{10}{3}-2\delta\right)\rho^3+\left(\dfrac{4}{3}+\dfrac{1}{4}\delta\right)\rho a^2\right]\dfrac{\cos^2\psi}{R^2} + \left[\left(4-\dfrac{9}{4}\delta\right)\rho^2+\left(\dfrac{3}{4}\delta\right)a^2\right]\dfrac{\cos\psi}{R} + \left[\left(\dfrac{1}{3}-\dfrac{1}{8}\delta\right)\rho^3-\left(2-\dfrac{5}{8}\delta\right)\rho a^2\right]\dfrac{1}{R^2}+2\rho\right\} \end{cases}$$

At the point of maximum stress, where $\rho = a$ and $\psi = 0$, the above reduce to

$$(17)\quad \begin{cases} \tau_{\rho\theta} = 0 \\[1ex] \left[\tau_{\psi\theta}\right]_{max.} = \dfrac{2PR}{\pi a^3}\left[1+\left(2-\dfrac{3}{4}\delta\right)\dfrac{a}{R}+\left(\dfrac{3}{2}-\dfrac{5}{8}\delta\right)\dfrac{a^2}{R^2}\right] \end{cases}$$

As is apparent from the foregoing development, further approximations utilizing additional terms in the stress function will result in extremely long and tedious calculations. This in itself is a limitation of this method. Therefore at this point, assuming the solution to be a rapidly converging one, we will stop and introduce actual values of the ratio of cross-sectional radius to the mean radius of curvature of the tore in order to compare results with other solutions.

Using these results in Equations (11), we may now write expressions representing a second approximation of the stress distribution in a cross-section of the incomplete torus.

$$\left.\begin{aligned} T_{\rho\theta} &= -\frac{PR}{\pi a^4}\left[\frac{3}{4}\zeta\left(\rho^2-a^2\right)\frac{\sin\psi}{R}+\left(\frac{2}{3}+\frac{1}{8}\zeta\right)\frac{\sin 2\psi}{R^2}\right] \quad \text{where } \zeta=\left[\frac{1-\frac{37}{72}\frac{a^2}{R^2}}{1-\frac{43}{192}\frac{a^2}{R^2}}\right] \\ T_{\psi\theta} &= \frac{PR}{\pi a^4}\left\{\left[\left(\frac{10}{3}-2\zeta\right)\rho^3+\left(\frac{4}{3}+\frac{1}{4}\zeta\right)\rho a^2\right]\frac{\cos^2\psi}{R^2}+\left[\left(4-\frac{9}{4}\zeta\right)\rho^2+\left(\frac{3}{4}\zeta\right)a^2\right]\frac{\cos\psi}{R}+\left[\left(\frac{1}{3}-\frac{1}{8}\zeta\right)\rho^3-\left(2-\frac{5}{8}\zeta\right)\rho a^2\right]\frac{1}{R^2}+2\rho\right\} \end{aligned}\right\} \text{—(16)}$$

At the point of maximum stress, where $\rho = a$ and $\psi = 0$, the above reduces to

$$\left.\begin{aligned} T_{\rho\theta} &= 0 \\ \left[T_{\psi\theta}\right]_{max} &= \frac{2PR}{\pi a^3}\left[1+\left(2-\frac{3}{4}\zeta\right)\frac{a}{R}+\left(\frac{3}{2}-\frac{5}{8}\zeta\right)\frac{a^2}{R^2}\right] \end{aligned}\right\} \text{——(17)}$$

As is apparent from the foregoing development, further approximations utilizing additional terms in the stress function will result in extremely long and tedious calculations. This in itself is a limitation of this method. Therefore at this point, assuming the solution to be a rapidly converging one, we will stop and introduce actual values of the ratio of cross-sectional radius to the mean radius of curvature of the torus in order to compare results with other solutions.

RESULTS

The distribution of shearing stress on a horizontal diameter is shown in Fig. 5 and the circumferential stress distribution in Fig. 6. In both cases a ratio of R/a equal to 4 is used since this realizes the worst condition (i.e. greatest curvature for a given cross-section) of any practical significance.

The quantity S appearing as the ordinate in both curves is a dimensionless quantity and is equal to $\frac{\tau_{\psi\theta}\pi a^2}{P}$, since $\tau_{\rho\theta}$ vanishes on both a horizontal diameter and the periphery. A stress distribution representing pure torsion of a straight circular shaft is shown by a dotted line in both figures. It is seen that the maximum stress actually existing in the tore is considerably greater than that derived from ordinary torsion theory. Both curves are in good agreement with similar ones derived from the exact solution by Frieberger. Points from Frieberger's curves appear as the small circles in Fig. 5 and 6.

In comparing the results of this solution with others, namely Göhner, Wahl, and Frieberger, the point of maximum stress will be used as a reference with different values of R/a. Table 1 gives values of K in the expression $\left[\tau_{\psi\theta}\right]_{max} = \frac{2PR}{\pi a^3}\left[K\right]$ for the several solutions.

Table 1.

R	Exact	This Solution		Other Approx. Solutions	
a	Frieberger	1st Approx.	2nd Approx.	Göhner	Wahl
4	1.376	1.313	1.371	1.372	1.400
5	1.293	1.250	1.287	1.295	1.310
6	1.237	1.209	1.234	1.239	1.252
8	1.171	1.156	1.171	1.172	1.184
10	1.136	1.125	1.134	1.135	1.145

RESULTS

The distribution of bending stress on a horizontal diameter is shown in Fig. 5 and the circumferential stress distribution in Fig. 6. In both cases a value of R/a equal to 4 is used since this requires the worst condition (i.e. greatest curvature for a given cross-section) of any practical significance.

The quantity K appearing as the ordinate in both curves is a dimensionless quantity and is equal to $\frac{T_{\psi\theta}\pi a^2}{P}$; since $T_{\psi\theta}$ vanishes on both a horizontal diameter and the periphery. A stress distribution corresponding pure torsion of a straight circular shaft is shown by a dashed line in both figures. It is seen that the maximum stress actually existing in the coil is considerably greater than that derived from ordinary torsion theory. Both curves are in good agreement with similar ones derived from the exact solution by Freiberger. Points from Freiberger's curves appear as the small circles in Fig. 5 and 6.

In comparing the results of this solution with others, namely Göhner, Wahl, and Freiberger, the point of maximum stress will be used as a reference with different values of R/a. Table 1 gives values of K in the expression $[T_{\psi\theta}]_{max} = \frac{2PR}{\pi a^3}\left[K\right]$ for the several solutions.

Table 1.

R/a	Exact Freiberger	This solution 1st approx.	This solution 2nd approx.	Other Approx. Solutions Göhner	Other Approx. Solutions Wahl
4	1.379	1.313	1.371	1.376	1.400
5	1.293	1.250	1.287	1.295	1.310
6	1.237	1.209	1.231	1.239	1.253
8	1.171	1.156	1.171	1.173	1.184
10	1.136	1.125	1.134	1.135	1.145

Table 1. indicates that the energy method applied to this problem produces results which compare favorably with other solutions. It also appears that the solution converges rapidly, since only five terms were used in the stress function.

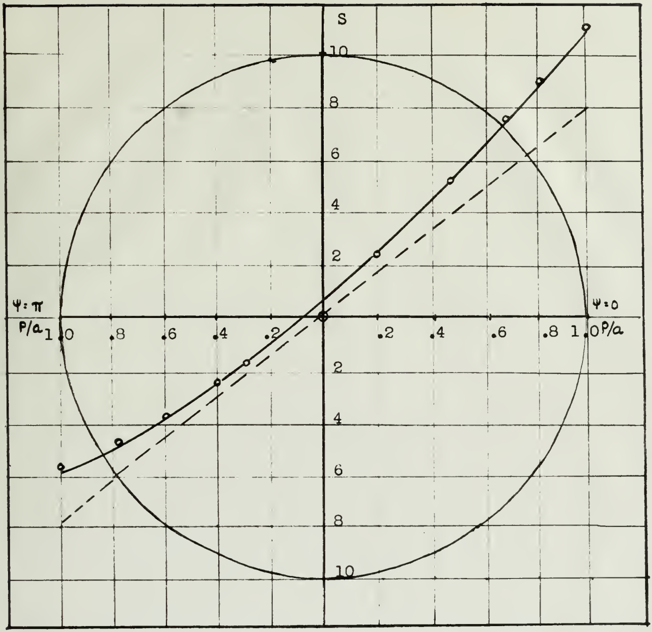

Fig. 5.

Distibution of shearing stress on a horizontal diameter for a/R=1/4. ($\psi = 0, \pi$). $S = (\tau_{\psi\theta})(\pi a^2)/P$. Frieberger's points are indicated by small circles.

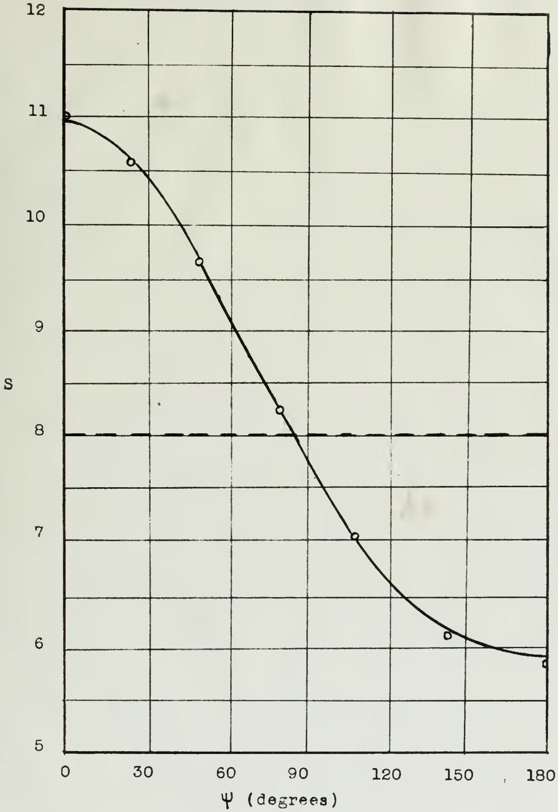

Fig. 6.

Circumferential stress distribution for $a/R=1/4$. $S=(\tau_{\psi\theta})(\pi a^2)/P$. Frieberger's points are indicated by small circles.

CONCLUSIONS

In discussing any conclusions from this investigation, it would be appropriate to recall the two questions that prompted it.

(1) Can the problem be solved by this method, and how do the results compare with those of other solutions?

(2) Does the problem particularly lend itself to solution by energy methods?

First, the method will work and acceptable results are obtained with a relatively few terms in the stress function. This is in itself worthy of note, since it allows a very complex problem to be attacked by the more elementary methods of mathematics.

However, in reference to the second question, there are limitations both inherent in the energy method and peculiar to this particular application, that strongly indicate the problem is not especially adapted to a solution by energy methods.

The energy method, except in unusual circumstances, does not provide an exact solution. Consequently, in the absence of an exact solution, there is no real basis for judging the results. The fact also that the energy method requires minimizing an integral, which is done only with extreme difficulty with any number of terms in the stress function, is a limitation to its adaptability.

In conclusion then, it may be said that this solution has the value of arriving at very good results using a relatively uncomplicated stress function of only five terms.

CONCLUSIONS

In discussing any conclusions from this investigation, it would be appropriate to recall the two questions that prompted it.

(1) Can the problem be solved by this method, and how do the results compare with those of other solutions?

(2) Does the problem particularly lend itself to solution by energy methods?

First, the method will work and acceptable results are obtained with a relatively few terms in the stress function. This is in itself worthy of note, since it allows a very complex problem to be attacked by the more elementary methods of mathematics.

However, in reference to the second question, there are limitations both inherent in the energy method and peculiar to this particular application, that strongly indicate the problem is not especially adapted to a solution by energy methods.

The energy method, except in unusual circumstances, does not provide an exact solution. Consequently, in the absence of an exact solution, there is no real basis for judging the results. The fact also that the energy method requires minimizing an integral, which is done only with extreme difficulty with any number of terms in the stress function, is a limitation in its adaptability.

In conclusion then, it may be said that this solution has the value of arriving at very good results using a relatively uncomplicated stress function of only five terms.

REFERENCES

(1) MICHELL, J. H. "The uniform torsion and flexure of incomplete tores, with application to helical springs." Proc. Royal Soc. 31 : 130 (1899)

(2) GÖHNER, O. Review in "Theory of Elasticity", Timoshenko and Goodier. pp. 391-395.

(3) SHEPHERD, W. M. "On the stresses in close-coiled helical springs." Quart. Journ. Mech. and Appl. Math. Vol. 3, Part 4, (1950).

(4) WAHL, A. M. "Stresses in heavy closely coiled helical springs." Trans. ASME, Appl. Mech., May-August, (1929).

(5) SOUTHWELL, R. V. "Some practically important stress systems in solids of revolution." Proc. Royal Soc. A 180: 367 (1942).

(6) FRIEBERGER, W. "The uniform torsion of an incomplete tore." Australian Journ Scien. Research, A. Vol 2, No. 3, 307 (1949).

APPENDIX A

According to the principle of least work which is used in this solution, an exact stress function would require selecting from all functions that satisfy the boundary condition those which minimize the strain energy.

Since in general this procedure is too difficult, a limited number N of suitable functions was selected to determine an approximate stress function.

In choosing functions of ρ and ψ in $\phi = \sum_{i=0}^{N} \alpha_i \phi_i$, the first consideration was the boundary condition $\phi_i = 0$ when $\rho = a$. This condition was satisfied by taking each ϕ_i to contain the factor $(\rho^2 - a^2)$.

Then $\phi = (\rho^2 - a^2) \sum_{i=0}^{N} \alpha_i f_i(\rho, \psi)$

With rectangular coordinates (ξ, η) in mind, where $\xi = \rho\cos\psi$ and $\eta = \rho\sin\psi$, the next logical step was to express $\sum f_i(\xi, \eta)$ in a power series. The first six terms of such a series were considered, namely those involving

$$1,\ \xi,\ \eta,\ \xi^2,\ \eta^2,\ \xi\eta$$

Since a horizontal diameter ($\eta = 0$) on a plane cross-section (θ is a constant) is an axis of symmetry for the ϕ surface, ϕ must be even in η and not contain terms involving odd powers of η. In general ξ will appear to all powers since ($\xi = 0$) is not axis of symmetry. Therefore the remaining terms expressed as functions of ρ and ψ are

$$1,\ \rho\cos\psi,\ \rho^2\cos^2\psi,\ \rho^2\sin^2\psi$$

The term $\frac{a^2}{R^2}$ in ϕ_4 while not consistent with this line of reasoning, appeared as a result of the binomial expansion used in approximating the stresses. It was extracted from Göhners solution where a like approximation was used.

APPENDIX A

According to the principle of least work which is used in this solution, an exact stress function would require selecting from all functions that satisfy the boundary condition those which minimize the strain energy.

Since in general this procedure is too difficult, a limited number N of suitable functions was selected to determine an approximate stress function.

In choosing functions of ρ and ψ in $\phi = \sum_{i=0}^{n} \alpha_i \phi_i$, the first consideration was the boundary condition $\phi_i = 0$ when $\rho = a$. This condition was satisfied by taking each ϕ_i to contain the factor $(\rho^2 - a^2)$.

Then $\phi = (\rho^2 - a^2) \sum_{i=0}^{n} \alpha_i f_i(\rho, \psi)$

With rectangular coordinates (ξ, η) in which, where $\xi = \rho \cos \psi$ and $\eta = \rho \sin \psi$, the next logical step was to express $\sum f_i(\xi, \eta)$ in a power series. The first six terms of such a series were considered, namely those involving

$$1, \quad \xi, \quad \eta, \quad \xi^2, \quad \eta^2, \quad \xi\eta$$

Since a horizontal diameter ($\eta = 0$) on a plane cross-section (θ is a constant) is an axis of symmetry for the ϕ surface, ϕ must be even in η and not contain terms involving odd powers of η. In general ξ will appear to all powers since ($\xi = 0$) is not axis of symmetry. Therefore the remaining terms expressed as functions of ρ and ψ are

$$1, \quad \rho \cos \psi, \quad \rho^2 \cos^2 \psi, \quad \rho^2 \sin^2 \psi$$

The term $\frac{a^2}{R^2}$ in ϕ_4 while not consistent with this line of reasoning, appeared as a result of the binomial expansion used in approximating the stresses. It was extracted from [illegible] solution where a like approximation was used.

Consequently the stress function was taken in the form

$$\phi = (\rho^2 - a^2)\left[\alpha_0 + \alpha_1\left(\frac{\rho}{R}\right)\cos\psi + \alpha_2\left(\frac{\rho}{R}\right)^2\cos^2\psi + \alpha_3\left(\frac{\rho}{R}\right)^2\sin^2\psi + \alpha_4\frac{a^2}{R^2}\right]$$

Here ρ was replaced by $\frac{\rho}{R}$ so that α_i would in all cases be the product of a dimensionless number and the factor $\frac{PR}{G\pi a^4}$.

In the first approximation the first two terms were used and in the second all five were introduced in ϕ.

Consequently the stress function was taken in the form

$$\phi = (\rho^2 - a^2)\left[\alpha_0 + \alpha_1\left(\frac{\rho}{R}\right)\cos\psi + \alpha_2\left(\frac{\rho}{R}\right)^2\cos^2\psi + \alpha_3\left(\frac{\rho}{R}\right)^2\sin^2\psi + \alpha_4\frac{a^2}{R^2}\right]$$

Here ρ was replaced by $\frac{\rho}{R}$ so that α_i would in all cases be the product of a dimensionless number and the factor $\frac{PR}{G\pi a^4}$.

In the first approximation the first two terms were used and in the second all five were introduced in ϕ.

APPENDIX B

The appearance of the term $\frac{1}{(R-\rho\cos\psi)^2}$ in the exact equations relating the stresses and the stress function gives rise to the occurrence of integrals of the type $\int_0^a\int_0^{2\pi} \frac{\rho^m \sin^n\psi \cos^q\psi \, d\rho \, d\psi}{(R-\rho\cos\psi)^2}$ and $\int_0^a\int_0^{2\pi} \frac{\rho^m \sin^n\psi \cos^q\psi \, d\rho \, d\psi}{(R-\rho\cos\psi)^3}$ in evaluating the strain energy and in consideration of the conditions of static equilibrium.

Taking the simplest form of the first type, where $m=1$, $n=0$ and $q=0$ we have

$$\int_0^a\int_0^{2\pi} \frac{\rho \, d\rho \, d\psi}{(R-\rho\cos\psi)^2}$$

Integrating first with respect to ψ and setting $R = c$ and $-\rho = b$

$$\int \frac{d\psi}{(c+b\cos\psi)^2}$$

Letting

$$P = \frac{\sin\psi}{(c+b\cos\psi)^2}$$

Then

$$\frac{dP}{d\psi} = \frac{\cos\psi(c+b\cos\psi) + b(1-\cos^2\psi)}{(c+b\cos\psi)^2} = \frac{b+c\cos\psi}{(c+b\cos\psi)^2}$$

$$= \frac{b - \frac{c^2}{b} + \frac{c}{b}(c+b\cos\psi)}{(c+b\cos\psi)^2} = \frac{c}{b}\left(\frac{1}{c+b\cos\psi}\right) - \frac{c^2-b^2}{b}\left[\frac{1}{(c+b\cos\psi)^2}\right]$$

Multiplying by $d\psi$ and integrating

$$\int \frac{dP}{d\psi} d\psi = P = \frac{\sin\psi}{c+b\cos\psi} = \frac{c}{b}\int \frac{d\psi}{c+b\cos\psi} - \frac{c^2-b^2}{b}\int \frac{d\psi}{(c+b\cos\psi)^2}$$

$$\int \frac{d\psi}{c+b\cos\psi} = -\frac{b}{c^2-b^2}\left(\frac{\sin\psi}{c+b\cos\psi}\right) + \frac{c}{c^2-b^2}\int \frac{d\psi}{c+b\cos\psi}$$

$$= \frac{1}{\sqrt{c^2-b^2}} \cos^{-1}\left(\frac{b+c\cos\psi}{c+b\cos\psi}\right) \quad \text{where } c^2 > b^2$$

APPENDIX B

The appearance of the term $\frac{1}{(R-\rho\cos\psi)^2}$ in the exact equations relating the stresses and the stress function gives rise to the occurrence of integrals of the type $\int_0^a\int_0^{2\pi}\frac{\rho^m\sin^n\psi\cos^g\psi\,d\rho\,d\psi}{(R-\rho\cos\psi)^2}$ and $\int_0^a\int_0^{2\pi}\frac{\rho^m\sin^n\psi\cos^g\psi\,d\rho\,d\psi}{(R-\rho\cos\psi)^3}$ in evaluating the strain energy and in consideration of the conditions of static equilibrium.

Taking the simplest form of the first type, where $m=1$, $n=0$ and $g=0$ we have,

$$\int_0^a\int_0^{2\pi}\frac{\rho\,d\rho\,d\psi}{(R-\rho\cos\psi)^2}$$

Integrating first with respect to ψ and setting $R=c$ and $-\rho=b$

$$\int\frac{d\psi}{(c+b\cos\psi)^2}$$

Letting

$$P=\frac{\sin\psi}{(c+b\cos\psi)}$$

Then

$$\frac{dP}{d\psi}=\frac{\cos\psi\,(c+b\cos\psi)+b\,(1-\cos^2\psi)}{(c+b\cos\psi)^2}=\frac{b+c\cos\psi}{(c+b\cos\psi)^2}$$

$$=\frac{b-\frac{c^2}{b}+\frac{c}{b}(c+b\cos\psi)}{(c+b\cos\psi)^2}=\frac{c}{b}\left(\frac{1}{c+b\cos\psi}\right)-\frac{c^2-b^2}{b}\left[\frac{1}{(c+b\cos\psi)^2}\right]$$

Multiplying by $d\psi$ and integrating

$$\int\frac{dP}{d\psi}d\psi=P=\frac{\sin\psi}{c+b\cos\psi}=\frac{c}{b}\int\frac{d\psi}{c+b\cos\psi}-\frac{c^2-b^2}{b}\int\frac{d\psi}{(c+b\cos\psi)^2}$$

$$\int\frac{d\psi}{(c+b\cos\psi)^2}=-\frac{b}{c^2-b^2}\left(\frac{\sin\psi}{c+b\cos\psi}\right)+\frac{c}{c^2-b^2}\int\frac{d\psi}{c+b\cos\psi}$$

$$=\frac{1}{\sqrt{c^2-b^2}}\cos^{-1}\left(\frac{b+c\cos\psi}{c+b\cos\psi}\right)\quad\text{where } c^2>b^2$$

Therefore

$$\int \frac{d\psi}{(c+b\cos\psi)^2} = -\frac{b}{c^2-b^2}\left(\frac{\sin\psi}{c+b\cos\psi}\right) + \frac{c}{(c^2-b^2)^{3/2}}\cos^{-1}\left[\frac{b+c\cos\psi}{c+b\cos\psi}\right]$$

Introducing the limits 0 and 2π, this reduces to

$$\frac{2\pi c}{(c^2-b^2)^{3/2}} \qquad \text{where} \qquad \begin{array}{l} c = R \\ b = -\rho \end{array}$$

Therefore

$$\int_0^a \int_0^{2\pi} \frac{\rho\, d\rho\, d\psi}{(R-\rho\cos\psi)^2} = 2\pi R \int_0^a \frac{\rho\, d\rho}{(R^2-\rho^2)^{3/2}} = 2\pi R \left[\frac{1}{(R^2-\rho^2)^{1/2}}\right]_0^a$$

$$= 2\pi \left[\frac{R}{(R^2-a^2)^{1/2}} - 1\right]$$

The other more complicated forms where $n \neq 0$ and $\zeta \neq 0$ are integrable in finite terms by similar reduction methods, but it is apparent that the work becomes excessively involved. Also the results in the form just developed are not readily usable in evaluating the unknown coefficients in the stress function.

In view of the foregoing, despite the fact that it was not actually necessary, it was expedient to approximate the stresses in such a manner that the integration was simplified and the results put in a usable form.

This device of approximating the stress equations compromised the requirement that the stresses satisfy the equations of equilibrium. However, it appears that, since the stresses do satisfy the conditions of minimum strain energy and static equilibrium, and give satisfactory results, the compromise may be tolerated.

Therefore

$$\int \frac{d\psi}{(c + b\cos\psi)^2} = -\frac{b}{c^2 - b^2}\left(\frac{\sin\psi}{c + b\cos\psi}\right) + \frac{c}{(c^2 - b^2)^{3/2}}\cos^{-1}\left[\frac{b + c\cos\psi}{c + b\cos\psi}\right]$$

Introducing the limits 0 and 2π, this reduces to

$$\frac{2\pi c}{(c^2 - b^2)^{3/2}} \qquad \text{where} \qquad \begin{aligned} c &= R \\ b &= -\rho \end{aligned}$$

Therefore

$$\int_0^a \int_0^{2\pi} \frac{\rho\, d\rho\, d\psi}{(R - \rho\cos\psi)^2} = 2\pi R \int_0^a \frac{\rho\, d\rho}{(R^2 - \rho^2)^{3/2}} = 2\pi R\left[\frac{1}{(R^2 - \rho^2)^{1/2}}\right]_0^a$$

$$= 2\pi\left[\frac{R}{(R^2 - a^2)^{1/2}} - 1\right]$$

The other more complicated forms where $n \neq 0$ and $\xi \neq 0$ are integrable in finite terms by similar reduction methods, but it is apparent that the work becomes excessively involved. Also the results in the form just developed are not readily usable in evaluating the unknown coefficients in the stress function.

In view of the foregoing, despite the fact that it was not entirely necessary, it was expedient to approximate the stresses in such a manner that the integration was simplified and the results put in a usable form.

This device of approximating the stress equations compromised the requirement that the stresses satisfy the equations of equilibrium. However, it appears that, since the stresses do satisfy the conditions of minimum strain energy and static equilibrium, and give satisfactory results, the compromise may be tolerated.